General Aggregation of Demand and Cost Sharing Methods

Tijs, S.H.; Koster, M.A.L.

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STEF TIJS

MAURICE KOSTER

Department of Econometrics
and
CentER for Economic Research
Tilburg University
PO Box 90153
NL - 5000 LE Tilburg
The Netherlands

e-mail: s.h.tijs@kub.nl and koster@kub.nl

Abstract: This paper extends the notion of cost sharing to models with general demand aggregation rules. In the process aggregated serial cost sharing mechanisms are defined and characterized. A framework for a dynamic view on cost sharing is provided, introducing the notion of consistency to the generalized cost sharing model. Corresponding optimistic and pessimistic cooperative cost games are defined and their cores are studied. In particular we show that the class of bankruptcy problems can be seen as a special class of cost sharing problems. It is seen that the serial mechanism in this specific case is closely related to the Constrained Equal Award rule.

1 Introduction

There are many real-life situations where individuals or different groups of individuals work together in a joint project. Cost sharing problems may arise when the joint project involves joint costs. The problem consists of determining "fair" shares for each of the participants such that total costs are cleared. Examples include determining cost shares for production of private or public goods using a jointly owned production facility, the allocation of joint overhead costs of a firm among its divisions (Shubik [1962]), and charging individuals for the use of a common network, airport or transit system (Young [1994]). In the literature, many cost sharing problems have been studied and various solution concepts have been introduced. Eligible candidates for cost sharing rules are those which can be characterized by a set of appropriate normative or game-theoretical properties.

First, we will focus on situations where a fixed group of agents is the owner of a joint production facility for a single (divisible) private good. A cost function c summarizes the costs for each level of production. We consider the case where the agents are allowed to consume different quantities of the good. Suppose that each of the agents has a particular demand for the good that is announced simultaneously. Then an amount of the good has to be produced such that all agents can be completely satisfied. The costs involved with this level of production have to be shared. Suppose that agent i's demand is given by q_i and that the minimal satisfactory level of production is determined by the aggregate $\sum_i q^i$ of the demands. Then costs $c(\sum_i q^i)$ have to be shared. A popular approach to the problem is to charge unit cost, so that agent i's cost share becomes $\xi_i(c, q^N) = \frac{q^i}{\sum_k q^k} \cdot c(\sum_k q^k)$ (Billera and Heath [1982], Tauman [1988], Young [1994]). Moulin and Shenker [1992] show that this device has its drawbacks when returns of production vary with the level of production. The main criticism on the *average cost sharing mechanism* in these situations is that it holds different types of agents equally responsible for the average cost. One could argue that if the returns of the technology are decreasing, small agents should be held less responsible for the imposed negative externality than the larger agents. Moulin and Shenker [1992] propose the *serial cost sharing mechanism* that is coherent with this idea. Besides several important normative aspects, the rule has excellent strategic features which make the mechanism compelling in congested situations.

In this paper we illustrate that techniques and ideas of cost sharing mechanisms can be used to study more general cost sharing models. In Section 2 we start by considering situations in which the total demand is not necessarily determined by the sum of the individual demands. For example, Moulin [1993] considers a production model of an excludable good where the generated level of output only depends on the maximum of the individual demands. It singles out a cost sharing mechanism that preserves the characteristics of the serial cost sharing mechanism, while the formulas are almost identical. We focus on defining and characterizing serial cost sharing mechanisms to situations where other demand aggregation rules may be used. A notion of weak consistency is developed for such situations. Furthermore we argue by the discussion on weak consistency that the usefulness and intuition behind characterizing properties for the cost sharing mechanisms for the model may heavily depend on the properties of the different aggregation rules. We characterize a class of demand aggregation rules that are nicely behaved with respect to the consistency concept.

Section 3 discusses a dynamical context in which cost sharing models for general aggregation of demands can be placed. In the first sections sharing costs was considered as a one shot happening. Now we adopt an approach similar to Aumann-Shapley pricing in order to study mechanisms that allocate marginal costs while the aggregate demand for output is processed. This leads to a new class of cost sharing mechanisms, large enough to contain many familiar cost sharing mechanisms like the generalized average cost mechanism and the aggregated serial rule and the reverse aggregated serial rule.

Though it is not immediately clear what the consistency concept should be in Section 2 and our proposal may look somewhat artificial, the transparency of the consistency concept that we propose in Section 4 for dynamical cost sharing is surprising. Furthermore we show that many well-known cost sharing mechanisms are consistent in the dynamic context. For example, among them are the (reverse) aggregate serial rule and the generalized proportional rule.

In Section 5 we consider the relation between cooperative game theory and dynamic cost sharing. We define and study the optimistic and pessimistic cost game; these (dual) games arise from dynamic cost sharing in a natural way. It turns out that the pessimistic cost game is concave. Furthermore its core is quite large and contains for instance the outcomes of the cost sharing mechanisms like the aggregate serial rule and the generalized proportional rule.

Finally, in Section 6 we focus on the relation between cost sharing problems and bankruptcy problems. It turns out that each bankruptcy problem gives rise to a cost sharing problem, where the cost function relates the level of the claims to the deficit that has to be shared. It turns out that the *Constraint Equal Award* rule for bankruptcy problems is closely related to the serial cost sharing mechanism for the associated cost sharing problem. The reverse serial mechanism is in a similar way related to the *Constraint Equal Loss* rule. The core of the pessimistic game associated with any cost sharing problem that can be derived from a bankruptcy situation is characterized.

Section 7 contains some final remarks.

2 Demand aggregation rules and serial cost sharing

Much of the existing cost sharing literature deals with situations where the cost of running a common production facility is related to the total demand estimated as the sum of the individual demands. Real-life situations may however give rise to a different way of aggregating the individual demands. For example, consider the construction of a landing strip for an airport. The length of the strip and thereby the costs for construction depend only on the largest planes that will use this strip for landing and take off. The agents are the companies and agent i's demand is the size of the agent's largest plane that is going to use the strip. In this case the total demand is determined by the largest plane of the agents. There is a vast and growing literature on *airport situations* (Littlechild and Owen [1973], Littlechild and Thompson [1977], Sudhölter and Potters [1995]), where the demands can be ordered and the maximum w.r.t. this ordering determines the level of output, and thereby the total costs of the project. Moulin [1993] considers the most natural situation among them, where the demands are (positive) real numbers and the natural complete ordering on $\mathbb{R}_+$ is used. Moulin provides a cost sharing mechanism for these situations that is to a large extent comparable with the serial cost sharing mechanism as in Moulin and Shenker [1992]; it shares the same normative and strategic characteristics and the formulas and techniques used are almost identical. Therefore we will also refer to it as a *serial cost sharing mechanism.* Furthermore, the serial outcome is a special one: it coincides with the Shapley value for the corresponding cost game (Dubey [1982]). In this section we extend the serial cost sharing mechanism to cost sharing mechanisms for situations where other types of aggregation rules may be used.

An *aggregation rule* can be seen as a mapping $\alpha : \bigcup_{N\subset \mathbb{N}; |N|<\infty} \mathbb{R}_+^N \to \mathbb{R}_+$, associating to each finite set of agents $N \subset \mathbb{N}$ with corresponding demand profile $q^N = (q^i)_{i\in N} \in \mathbb{R}_+^N$, the total demand $\alpha(q^N) \in \mathbb{R}_+$ as the minimum level of output that is needed in order to satisfy all agents of N. Given a demand profile $q^N \in \mathbb{R}_+^N$, $i \in N$ and $q \in \mathbb{R}_+$, we define (q^N_{-i}, q) as the demand profile out of q^N according to which only agent i's demand is interchanged with q. For $i \in N$ we denote by $q^{i,N}$ the demand profile out of q^N such that the demands of agents $j \in N$ is set to q^i whenever $q^j \geq q^i$. For instance, for $N = \{1, 2, 3\}$ and $q^N = (1, 2, 3)$, the profile $q^{2,N}$ is equal to $(1, 2, 2)$.
The set of all subsets of N is denoted by $\mathcal{P}(N)$. A demand profile $x \in \mathbb{R}_+^N$ is said to be (weakly) smaller than the demand profile $y \in \mathbb{R}_+^N$ iff $x_i \leq y_i$ for all $i \in N$. We denote this by $x \leq y$.

We will now fix the aggregation rule α. For the moment, we will only assume that if the agents (weakly) increase their demands the aggregate demand (weakly) increases: $x \leq y \Rightarrow \alpha(x) \leq \alpha(y)$. The space of all such aggregation rules will be denoted by $\mathcal{A}$. Define $\mathcal{A}^0 := \{\alpha \in \mathcal{A} \,|\, \alpha(0) = 0\}$. It is the subclass of $\mathcal{A}$ consisting of all those aggregation rules having the property that nothing has to be produced when there is no demand at all.

Example 2.1 Consider $\alpha_{\max} \in \mathcal{A}^0$ defined by $\alpha_{\max}(q^N) = \max_{i \in N} q^i$. It is the aggregation rule used for the problem of constructing an airstrip where agents have a particular demand for it's length.

Example 2.2 Consider a production facility for a certain type of cars. When an agent's demand expresses the number of cars he want to have, then $\alpha_{\text{sum}}(q^N) := \sum_{i \in N} q^i$ is the corresponding aggregation rule.

Example 2.3 Consider a common production facility for which there is a loss of output during production depending on the largest demander among the agents of N. In order to reach the minimal satisfactory level of production this loss has to be compensated. In this case the aggregation may take the form $\alpha(q^N) := \alpha_{\text{sum}}(q^N) + \lambda(\alpha_{\max}(q^N))$. Here λ is the function that determines the compensation.

Example 2.4 Consider the central administration of a multi-divisional firm. Suppose the demand for service by the different divisions can be expressed in time (hours). Then the aggregation may take the form $\alpha(q^N) = \alpha_{\text{sum}}(q^N) + q$, where q is the demand of the central administration for own services (independent from the other demands). Note that $\alpha \in \mathcal{A} \backslash \mathcal{A}^0$ iff $q > 0$.

Example 2.5 Suppose that a production facility for wall paper can produce only rolls of certain length, say 1. This situation gives rise to the aggregation rule $\alpha \in \mathcal{A}^0$ given by $\alpha(q^N) = \lceil \alpha_{\text{sum}}(q^N) \rceil (:= \min \{n \in \mathbb{N} \mid n \geq \alpha_{\text{sum}}(q^N)\})$.

We will concentrate on situations where there is given a cost function that summarizes information about cost incurring from production of any level of output. Especially, a cost function is a mapping $c : \mathbb{R}_+ \to \mathbb{R}_+$. Furthermore we will require that a cost function is non-decreasing and that there are no fixed costs to be shared, i.e. $c(0) = 0$. The space of all those cost functions is denoted by $\mathcal{C}$. Suppose the agents announce their demands as described by the demand profile q^N. The problem of sharing the costs $c(\alpha(q^N))$ by the agents of N will be denoted by the ordered triple $\langle c, \alpha, q^N \rangle \in \mathcal{C} \times \mathcal{A} \times \mathbb{R}^N_+$. Let $\mathcal{G}$ denote the space of all such cost sharing problems. A *cost sharing mechanism* is a mapping $\xi : \mathcal{G} \to \bigcup_{N \subset \mathbb{N}; |N| < \infty} \mathbb{R}^N_+$ that relates each cost sharing problem $\langle c, \alpha, q^N \rangle$ to a vector of cost shares $\xi(c, \alpha, q^N) \in \mathbb{R}^N_+$. We will focus on those cost sharing mechanisms that are *efficient* (EFF), i.e. mechanisms that distribute exactly total costs $c(\alpha(q^N))$ for production among the agents.
If ξ allocates costs in such a way that for all cost sharing problems $\langle c, \alpha, q^N \rangle$ any agent's cost share is non-decreasing in the size of his demand, i.e. the mapping $q \mapsto \xi_i(c, \alpha, (q^N_{-i}, q))$ is non-decreasing, then ξ is called *demand monotonic* (DM).

Example 2.6 Consider the cost sharing problem $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ where $N := \{1, 2, \ldots, n\}$ for certain $n \in \mathbb{N}$. Without loss of generality assume that $q^1 \leq q^2 \leq \ldots \leq q^n$. The *serial cost sharing mechanism* ξ^{s} takes into account $|N| + 1$ intermediate production stages $x_0^\alpha, x_1^\alpha, \ldots, x_n^\alpha \in [0, \alpha(q^N)]$, defined by

$$\begin{aligned} x_0^\alpha &= 0 \\ x_i^\alpha &= \alpha(q^{i,N}) \quad \text{for } i \in N \end{aligned}$$

The serial cost sharing mechanism ξ^{s} is now defined for $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ by

$$\xi_i^{\mathrm{s}}(c, \alpha, q^N) := \sum_{k=1}^{i} \frac{c(x_k^\alpha) - c(x_{k-1}^\alpha)}{n + 1 - k} \quad \text{for all } i \in N$$

Clearly, this is the serial rule as in Moulin and Shenker [1992] resp. Moulin [1993] when $\alpha = \alpha_{\mathrm{sum}}$ resp. $\alpha = \alpha_{\mathrm{max}}$. For general $N \subset \mathbb{N}$ of cardinality $n \in \mathbb{N}$, let $f : N \to \{1, 2, \ldots, n\}$ be a bijection ordering the agents, such that $f(i) \leq f(j)$ iff $q^i \leq q^j$. A formula for the serial mechanism for any cost sharing problem $\langle c, \alpha, q^N \rangle$ is then obtained by replacing i by $f(i)$ in the above formulas for ξ^s.

Example 2.7 Consider the cost sharing problem $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ with $N := \{1, 2, \ldots, n\}$ for certain $n \in \mathbb{N}$. Without loss of generality assume that $q^1 \leq q^2 \leq \ldots \leq q^n$. Let $(x_k^\alpha)_{k \in N \cup \{0\}}$ be the profile of intermediate production stages defined in the former example. The *reverse serial cost sharing mechanism* takes into account $|\,N\,| + 1$ intermediate production stages $\bar{x}_0^\alpha, \bar{x}_1^\alpha, \ldots, \bar{x}_n^\alpha \in [0, \alpha(q^N)]$ defined by $\bar{x}_k^\alpha := \alpha(q^N) - x_k^\alpha$ for all $k \in \{0, 1, \ldots, n\}$. The reverse serial cost sharing mechanism, denoted by ξ^{rs}, is now defined by

$$\xi_i^{\mathrm{rs}}(c, \alpha, q^N) := \sum_{k=1}^{i} \frac{c(\bar{x}_k^\alpha) - c(\bar{x}_{k+1}^\alpha)}{n + 1 - k} \quad \text{for all } i \in N$$

A definition of the reverse serial cost sharing mechanism for general finite $N \subset \mathbb{N}$ is obtained in the same way as in the former example.

Example 2.8 For $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ the generalized proportional cost sharing mechanism ξ^{p} is given by

$$\xi_i^{\mathrm{p}}(c, \alpha, q^N) := \frac{q^i}{\sum_{k \in N} q^k} c(\alpha(q^N)) \quad \text{for all } i \in N$$

The cost sharing mechanisms of the examples 2.6, 2.7 and 2.8 are obviously efficient. Also do they treat agents equally in the sense that the cost shares of any pair of agents with demands of equal size are the same (*Equal Treatment of Equals* (ETE)). The mechanisms are demand monotonic and additive w.r.t. the cost function (ADD). We will now study the serial cost sharing mechanism in more detail. First we will adapt the normative

approach to serial cost sharing as in Moulin and Shenker [1992].
The serial mechanism has the *Independence of Size of Larger Demands* (ISLD) property, i.e. an agent's cost share does not depend on the size of the demands that are larger than his. More formally, a cost sharing mechanism ξ satisfies *Independence of Size of Larger Demands* (ISLD) if for all $c \in \mathcal{C}, \alpha \in \mathcal{A}$ and $i, j \in N$

$$\{\, q^j \geq q^i \;\text{ and }\; \tilde{q}^j \geq q^i \,\} \Rightarrow \xi_i(c, \alpha, q^N) = \xi_i(c, \alpha, (q^N_{-j}, \tilde{q}^j))$$

We can characterize the serial rule for general aggregation models by the properties EFF, ISLD and ETE:

Theorem 2.9 *ξ^s is the only efficient cost sharing mechanism satisfying the properties* ISLD *and* ETE.

Proof: It is straightforward to show that ξ^s satisfies ISLD and ETE. Suppose ξ is an efficient cost sharing mechanism satisfying ISLD and ETE. Take $c \in \mathcal{C}$ and rank the agents according to their demands. Without loss of generality we assume that $N = \{1, 2, \ldots, n\}$ and $q^1 \leq q^2 \leq \ldots \leq q^n$. Now the uniqueness proof can be given by using an induction argument. For agent 1:

$$\xi_1(c, \alpha, q^N) \;=\; \xi_1(c, \alpha, q^{i,N}) = \frac{c(x_1^\alpha)}{n} \;=\; \xi_1^s(c, \alpha, q^N)$$

The first equality follows by ISLD, while the second holds by a combination of ETE and EFF. Now take $i \in N$ and suppose that $\xi_j(c, \alpha, q^N) = \xi_j^s(c, \alpha, q^N)$ for $j < i$. Then if $(x_k^\alpha)_{k \in N}$ denotes the profile of all intermediate production stages according to aggregation rule α and profile q^N, then we have

$$\begin{aligned}
\xi_i(c, \alpha, q^N) \;&=\; \xi_i(c, \alpha, q^{i,N}) \;=\; \frac{1}{n+1-i}\Big(c(x_i^\alpha) - \sum_{k=1}^{i-1} \xi_k(c, \alpha, q^{i,N})\Big) \\
&=\; \frac{1}{n+1-i}\Big(c(x_i^\alpha) - \sum_{k=1}^{i-1} \xi_k^s(c, \alpha, q^{i,N})\Big) \;=\; \xi_i^s(c, \alpha, q^{i,N}) \\
&=\; \xi_i^s(c, \alpha, q^N)
\end{aligned}$$

We get the first and the last equality by ISLD, the second and the fourth by a combination of ETE and EFF, and the third by our induction hypothesis.

□

Remark 2.10 Suppose ξ is a cost sharing mechanism that satisfies ISLD and ETE on a subclass G of cost sharing problems in $\mathcal{G}$. The proof of Theorem 2.9 can be adopted to show that ξ coincides with ξ^s on G.

As has been done, one possibility to measure an agent's cost impact is by direct comparison of the size of the agents' demands. This means that agent i is considered to have less an impact on total costs than agent j if and only if $q^i \leq q^j$, no matter what the cost structure is. Another way is considering an agent's *unanimity cost*, i.e. the costs that would arise when all the individual demands equal the demand of the agent in question. Agent i has a smaller cost impact than agent j if and only if $\min\{c(\alpha(q^i, \ldots, q^i)), c(\alpha(q^N))\} \leq \min\{c(\alpha(q^j, \ldots, q^j)), c(\alpha(q^N))\}$. Here we take the minimum with total costs for production, since it is reasonable that an agent cannot be held responsible for production costs that have not been made. Using this new criterion, agent i is still considered to have less impact on total costs than agent j whenever $q^i \leq q^j$, since c is assumed to be non-decreasing. Suppose that the unanimity costs for agent i are zero, i.e. $c(\alpha(q^i, q^i, \ldots, q^i)) = 0$. One could argue that agent i should be charged no cost at all, since he attains the lowest possible level of cost impact, i.e. 0. We say that cost sharing mechanisms that distribute costs in accordance with this idea have the *Free Lunch* (FL) property. More formally,

Definition 2.11 A cost sharing mechanism ξ satisfies FL, if, for all cost sharing problems $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ it holds that for all $i \in N$ $c(\alpha(q^i, q^i, \ldots, q^i)) = 0$ implies $\xi_i(c, \alpha, q^N) = 0$.

In case the demands are aggregated by taking the sum, this is nothing but the Free Lunch property as in Moulin and Shenker [1994].

If agent $i \in N$ has the smallest demand and $c(\alpha(q^{i,N})) = c(\alpha(q^N))$, one could argue that agent i has the same impact on total costs as any other agent and therefore any in this respect reasonable cost sharing mechanism should treat each agent equally. When a cost sharing mechanism does so in fact, it is said to be *egalitarian* (EG). More formally,

Definition 2.12 A cost sharing mechanism satisfies EG, if, for all cost sharing problems $\langle c, \alpha, q^N \rangle \in \mathcal{G}$ it holds that $c(\alpha(q^{i,N})) = c(\alpha(q^N))$ for $i \in N$ with $q^i = \min\{q^k \mid k \in N\}$ implies $\xi_j(c, \alpha, q^N) = \frac{c(\alpha(q^N))}{|N|}$ for all $j \in N$.

Jansen and Peters [1995] discuss this property for the setting where the total demand is estimated as the sum of the individual demands.

The serial mechanism ξ^s satisfies both FL and EG. These properties together with a limited form of consistency lead to a second characterization of the serial cost sharing mechanism for certain types of demand aggregation that will be specified below.

For the moment let the aggregation rule be α_{sum}. Suppose a group of agents decide to leave and each of the leaving agents makes a pre-payment equal to his cost share as determined by the cost sharing mechanism ξ for the cost sharing problem $\langle c, \alpha, q^N \rangle$. Then the remaining group of agents, say $S \subset N$, is confronted with the *reduced cost sharing problem* where the aggregate demand of the leaving group has to be satisfied and their pre-payments have to be taken into account. This amounts to the following adaptation of the cost function c. First translate the cost function over $-\sum_{i\in N\setminus S} q^i$ w.r.t. the input argument, and lower total costs for all remaining production levels by the pre-payment of coalition S, i.e. $\sum_{i\in N\setminus S}\xi_i(c,\alpha,q^N)$). Set the translated function to 0 for those values of demand for which the translated function has a negative value, in order to achieve a cost function with positive values only. A cost sharing mechanism is *consistent* if it constitutes the same cost shares for the remaining agents in the reduced cost sharing problem $\langle c_S, \alpha, q^S\rangle$. Thomson [1996] provides the following definition (for $\alpha = \alpha_{\text{sum}}$).

Definition 2.13 A cost allocation mechanism ξ is called *consistent* if for all finite $N \subset \mathbb{N}$, $S \subseteq N$, demand profiles q^N and cost functions $c \in \mathcal{C}$, we have $\xi_i(c_S, \alpha_{\text{sum}}, q^S) = \xi_i(c, \alpha_{\text{sum}}, q^N)$ for all $i \in S$, where the cost function $c_S \in \mathcal{C}$ is defined by $c_S(q) := \max\{c(q + \sum_{i\in N\setminus S} q^i) - \sum_{i\in N\setminus S}\xi_i(c, \alpha_{\text{sum}}, q^N), 0\}$ whenever $q > 0$ and $c_S(0) := 0$.

The serial cost sharing mechanism is not consistent. Moulin and Shenker [1994] show that the serial rule for $\alpha = \alpha_{\text{sum}}$ is *weakly consistent*, in the sense that the mechanism constitutes the same cost shares for the agents of $S = N\setminus\{1\}$ in the reduced cost sharing problem $\langle c_S, \alpha_{\text{sum}}, q^S\rangle$ as in the original problem $\langle c, \alpha_{\text{sum}}, q^N\rangle$, where agent 1 is assumed to have the smallest demand. Again, the intuition behind this is that when agent 1 leaves with paying his cost share $\xi^{\text{s}}_1(c, \alpha_{\text{sum}}, q^N)$ the other agents are left with the situation where the demand q^1 has to be satisfied and the pre-payment $\xi^{\text{s}}_1(c, \alpha_{\text{sum}}, q^N)$ is taken into account in the calculation of the costs that are still to be covered by the agents of $N\setminus\{1\}$. Kolpin [1994] introduces *baseline consistency* for multi-commodity situations, which in fact is the natural extension of weak consistency to multi-commodity situations.

For airport problems a similar notion of consistency is developed, based on the same intuition. For the reduced situation where one of the agents with smallest demand has left and contributed his cost share, the reduced cost function $\bar{c}$ is constructed in a similar way by translating the original cost function c. A formal definition can be given by adapting the definition of weak consistency, wherein only the translation w.r.t. the input argument of c is left behind. Sudhölter and Potters [1995] show that though the Shapley value and thus the serial outcome for airport situations is not consistent, it is *weakly-consistent*. This property is the analogue of weak consistency for situations in which α_{sum} is the aggregation rule. In the same paper it is shown that weak consistency coincides with a limited form of *μ-consistency* and *ν-consistency*. Though for both weak-consistency concepts it appears to be unambiguous how to adapt the costs w.r.t. the pre-payed amount $\xi_1(c, \alpha, q^N)$ for $\alpha = \alpha_{\text{sum}}, \alpha_{\text{max}}$, it is not immediately clear what underlying idea motivates the use of different translations w.r.t. input argument of the cost function.

The concepts of weak consistency have in common that when the smallest agent, by assumption agent 1, leaves and pre-pays $\xi_1(c, \alpha, q^N)$ then the reduced cost function $\bar{c}$ is constructed by translating c w.r.t. the input argument over minus the marginal contribution of agent 1 to the aggregated demand for the good, i.e. $-(\alpha(q^N) - \alpha(q^{N\setminus\{1\}}))$ and decreasing the costs over the whole interval by $\xi_1(c, \alpha, q^N)$. In case $\alpha = \alpha_{\text{max}}$ resp. $\alpha = \alpha_{\text{sum}}$ this constitutes the reduced cost function by translating the cost function over

0 resp. $-q^1$ for the input argument, while in both situations total costs are lowered over the whole interval by $\xi_1(c, \alpha, q^N)$. We will use the marginal contribution of the smallest agent to the total demand in order to extend the meaning of weak consistency to situations in which other aggregation rules may be used.

Definition 2.14 Suppose $\mathcal{A}' \subset \mathcal{A}$. A cost sharing mechanism ξ is called *weakly consistent* (WCONS) on $\mathcal{A}'$ if for all finite $N \subset \mathbb{N}$, $(c, \alpha, q^N) \in \mathcal{C} \times \mathcal{A}' \times \mathbb{R}_+^N$ and $j \in N$ such that $q^j = \min\{q^k \mid k \in N\}$ we have $\xi_i(c, \alpha, q^N) = \xi_i(\bar{c}, \alpha, (q_{-j}^N, 0))$ for all $i \in N \backslash \{j\}$ and where $\bar{c} \in \mathcal{C}$ is given by $\bar{c}(q) = \max\{c(q + \alpha(q^N) - \alpha((q_{-j}^N, 0))) - \xi_j(c, \alpha, q^N), 0\}$ for all $q \in \mathbb{R}_+$.

The aggregation rules $\alpha_{\max}$ and α_{sum} have the property that for each agent $i \in N$ with the smallest demand his marginal contribution is invariant w.r.t. larger demands of other agents, i.e. the aggregation rule allows the bigger agents to vary their demands without affecting the smallest agent's marginal contribution to the aggregated demand profile as long as they maintain to be larger in the changed profile. For α_{sum} and $\alpha_{\max}$ the same holds for agents with the smallest *non-zero* demand. In other words,

Definition 2.15 Let $N \subset \mathbb{N}$ be a finite set of agents and define for $i \in N$ the set of profiles $P_i \subset \mathbb{R}_+^N$ according to which agent i has the one of the smallest non-zero demands, q^i. An aggregation rule $\alpha \in \mathcal{A}$ satisfies the *Weak Marginality Principle* (WMP) if for all $i \in N$, and $x^N, y^N \in P_i$ it holds that

$$\alpha(x^N) - \alpha(x_{-i}^N, 0) = \alpha(y^N) - \alpha(y_{-i}^N, 0)$$

Let $\mathcal{A}^*$ denote the subset of $\mathcal{A}$ of all aggregation rules satisfying WMP and $\mathcal{G}^* := \mathcal{C} \times \mathcal{A}^* \times \mathbb{R}_+^N$.

Example 2.16 Suppose $\alpha \in \mathcal{A}^*$. Then it holds that $\alpha(1,1,1) - \alpha(1,1,0) = \alpha(1,1,0) - \alpha(1,0,0)$.

Theorem 2.17 *ξ^s is the unique additive, efficient, weakly consistent, egalitarian cost sharing method, satisfying Free Lunch on $\mathcal{G}^*$.*

Proof: Without loss of generality assume that $N = \{1, 2, \ldots, n\}$ and q^N is an ordered demand profile, i.e. $q^1 \leq q^2 \leq \ldots \leq q^n$.
[1]) Clearly, ξ^s is additive and efficient. Furthermore, ξ^s is egalitarian. Suppose that for some cost function c, $c(\alpha(q^1, q^1, \ldots, q^1)) = c(\alpha(q^N))$. For all $k \in \{2, \ldots, n\}$ we

have $\alpha(q^{1,N}) \leq \alpha(q^{k,N}) \leq \alpha(q^N)$. So $c(\alpha(q^{k,N})) = c(\alpha(q^{1,N}))$. But then

$$\begin{aligned}\xi_i^s(c,\alpha,q^N) &= \sum_{k=1}^{i} \frac{c(\alpha(q^{k,N})) - c(\alpha(q^{k-1,N}))}{n+1-k} \\ &= \frac{c(\alpha(q^{1,N}))}{n} \\ &= \frac{c(\alpha(q^N))}{n}\end{aligned}$$

We prove that ξ^s is weakly consistent on $\mathcal{A}^*$ by induction on the number of agents. If $q^1 > 0$ then WMP implies $\alpha(q^{2,N}) - \alpha((q_{-1}^{2,N},0) = \alpha(q^N) - \alpha((q_{-1}^N,0))$. But of course this equality is valid if $q^1 = 0$. Then together with EG we have in both cases

$$\begin{aligned}\xi_2^s(\bar{c},\alpha,(q_{-1}^N,0)) &= \frac{\bar{c}(\alpha((q_{-1}^{2,N},0))}{n-1} \\ &= \frac{1}{n-1}(c(\alpha((q_{-1}^{2,N},0)) + \alpha(q^N) - \alpha((q_{-1}^N,0))) - \xi_1^s(c,\alpha,q^N)) \\ &= \frac{c(\alpha(q^{2,N}))}{n-1} - \frac{c(\alpha(q^{1,N}))}{(n-1)n} \\ &= \xi_2^s(c,\alpha,q^N)\end{aligned}$$

Now take $i \in N\backslash\{1,2\}$. Suppose that for $k \in \{1,2,\ldots,i-1\}$ it holds that

$$\xi_k^s(\bar{c},\alpha,(q_{-1}^N,0)) = \xi_k^s(c,\alpha,q^N)$$

Recapitulating the formulas of the serial mechanism we get

$$\begin{aligned}\xi_i^s(\bar{c},\alpha,(q_{-1}^N,0)) &= \textstyle\sum_{k=1}^{i} \frac{\bar{c}(\alpha((q_{-1}^{k,N},0)))-\bar{c}(\alpha(q^{k-1,N}))}{n+1-k} \\ &= \xi_{i-1}^s(\bar{c},\alpha,(q_{-1}^N,0)) + \tfrac{\bar{c}(\alpha((q_{-1}^{i,N},0)))-\bar{c}(\alpha((q_{-1}^{i-1,N},0)))}{n-i+1}\end{aligned}$$

By WMP we have $\alpha(q^N) - \alpha((q_{-1}^N,0)) = \alpha(q^{i,N}) - \alpha((q_{-1}^{i,N},0))$ and thus

$$\begin{aligned}\xi_i^s(\bar{c},\alpha,(q_{-1}^N,0)) &= \xi_{i-1}^s(c,\alpha,q^N) + \tfrac{c(\alpha((q_{-1}^{i,N},0))+\alpha(q^N)-\alpha((q_{-1}^N,0)))-c(\alpha((q_{-1}^{i-1,N},0))+\alpha(q^N)-\alpha((q_{-1}^N,0)))}{n-i+1} \\ &= \xi_{i-1}^s(c,\alpha,q^N) + \tfrac{c(\alpha(q^{i,N}))-c(\alpha(q^{i-1,N}))}{n-i+1} \\ &= \xi_i^s(c,\alpha,q^N)\end{aligned}$$

[2]) ξ^s is the only cost sharing mechanism with the properties EFF, FL, WCONS, EG and ADD on $\mathcal{G}^*$. Suppose ξ is a cost sharing mechanism satisfying the same combination of properties. We will show that in that case $\xi = \xi^s$.

For a given cost function c we define the cost functions $c_1, c_2 \in C$ by

$$\begin{cases} c_1(q): &= \max\{c(q) - c(\alpha(q^{1,N})), 0\} & \text{for all } q \in \mathbb{R}_+ \\ c_2(q): &= \min\{c(q), c(\alpha(q^{1,N}))\} & \text{for all } q \in \mathbb{R}_+ \end{cases}$$

Then $c = c_1 + c_2$. By ADD it suffices to show that $\xi_i(c_j, \alpha, q^N) = \xi_i^s(c_j, \alpha, q^N)$ for all $i \in N$ and $j \in \{1, 2\}$.
Now EG implies $\xi_i(c_2, \alpha, q^N) = \frac{1}{n}c(\alpha(q^{1,N})) = \xi_i^s(c_2, \alpha, q^N)$ for all $i \in N$.
We prove the remaining part by induction on the size of the group of agents. Suppose $|N| = 1$. Then by FL we have $\xi_1(c_1, \alpha, q^N) = 0 = \xi_1^s(c_1, \alpha, q^N)$. Now suppose that $\xi(c_1, \alpha, q^N) = \xi^s(c_1, \alpha, q^N)$ for $|N| = 1, \ldots, i-1$ for $i \geq 2$. Then from WCONS and $k \in \{2, 3, \ldots, i\}$

$$\xi_k(c_1, \alpha, q^N) = \xi_k(\bar{c}_1, \alpha, (q_{-1}^N, 0)) = \xi_k^s(\bar{c}_1, \alpha, (q_{-1}^N, 0)) = \xi_k(c_1, \alpha, q^N)$$

□

This characterization is only confined to those problems with an aggregation rule that satisfies WMP, as is illustrated in the next example.

Example 2.18 Suppose the three agents in $N = \{1, 2, 3\}$ face the cost sharing problem $\langle c, \alpha, q^N \rangle$ where $q^N = (1, 2, 3), \alpha(x) = (\sum_i x^i)^2$ and $c(q) = q^2$. It is clear that α does not satisfy WMP. So the total aggregated demand is $\alpha(1, 2, 3) = 36$ and for the reduced problem $\alpha(0, 2, 3) = 25$. It is straightforward to see that the cost function for the reduced situation where agent 1 has left and prepayed his serial cost share $\frac{1}{3}c(\alpha(1, 1, 1)) = 27$, is for all $q \in \mathbb{R}_+$ given by

$$\begin{aligned} \bar{c}(q) &= \max\{c(q + \alpha(1, 2, 3) - \alpha(0, 2, 3)) - \xi_1^s(c, \alpha, q^N), 0\} \\ &= \max\{c(q + 11) - 27, 0\} \\ &= (q + 11)^2 - 27 \end{aligned}$$

Thus $\xi_2^s(\bar{c}, \alpha, (q_{-1}^N, 0) = \frac{1}{2}\bar{c}(\alpha(0, 2, 2)) = \frac{1}{2}\bar{c}(16) = 341$, while $\xi_2^s(c, \alpha, q^N) = \xi_1^s(c, \alpha, q^N) + \frac{1}{2}(c(\alpha(1, 2, 2)) - c(\alpha(1, 1, 1))) = 27 + \frac{1}{2}(c(25) - c(9)) = 299$.
This means that the serial mechanism does not satisfy WCONS.

The cost sharing problems that we study here form a subclass of the multi-commodity situations that are studied in Friedman and Moulin [1995] and Sprumont [1996]. Our approach compensates for the loss of generality by fully exploiting the special structure of the cost sharing problem that is implied by the aggregation rule. Also, this gives us the opportunity to analyse classes of problems without placing further restrictive regularity assumptions on the associated cost functions. Cost sharing problems can be classified by different properties of the aggregation rules. So, analysing the properties of an aggregation rule will help us to understand the nature of different classes of cost sharing problems. Below, we characterize a specific class of aggregation rules, which can be seen as an example of such an analysis.

An aggregation rule is *symmetric* if no permutation of the agents' demands affect the minimal level of production at which the agents can be satisfied. Moreover, an aggregation rule $\alpha \in \mathcal{A}$ is *homogeneous* if $\alpha(\lambda x) = \lambda \alpha(x)$ for all $\lambda \in \mathbb{R}_+$. In particular, if α is homogeneous then $\alpha \in \mathcal{A}^0$. We will call a symmetric, homogeneous aggregation rule *regular* if agents with zero demand may be deleted from the group of agents N without changing the total aggregated demand, i.e. $q^i = 0$ implies $\alpha(q^N) = \alpha(q^{N\setminus\{i\}})$. For instance, $\alpha_{\max}$ and α_{sum} are both regular.

Each permutation $\sigma : N \to N$ defines a *comonotonic* set $C_N^\sigma := \{x \in \mathbb{R}_+^N \mid x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)}\}$. As we will show now, the regular aggregation rules having the WMP property take a very specific form and are in fact the aggregation rules that are linear and symmetric w.r.t. the comonotonic sets in $\cup_{N\subset\mathbb{N};|N|<\infty}\mathbb{R}_+^N$.

Proposition 2.19 *Let $\alpha \in \mathcal{A}^*$ be regular. Then there are $\beta, \gamma \in \mathbb{R}$ such that $\alpha = \beta\,\alpha_{\text{sum}} + \gamma\,\alpha_{\max}$.*

Proof: The proof is by induction. First, suppose $N \subset \mathbb{N}$ and $|N| = 2$. We may assume that $N = \{1, 2\}$. Then for $x \in \mathbb{R}_+^N$ with $x_1 \leq x_2$ it follows by WMP that $\alpha(x) = \alpha(x_1, x_1) - \alpha(0, x_1) + \alpha(0, x_2)$. Next, homogeneity of α implies $\alpha(x) = x_1\,(\alpha(1,1) - \alpha(0,1)) + x_2\,\alpha(0,1)$. Take $\beta := \alpha(1,1) - \alpha(0,1)$ and $\gamma := 2\,\alpha(0,1) - \alpha(1,1)$. Then $\alpha(x) = \beta \cdot (x_1 + x_2) + \gamma \cdot \max\{x_1, x_2\}$. Furthermore, α is symmetric, so $\alpha(0,1) = \alpha(1,0)$ and thus $\beta = \alpha(1,1) - \alpha(1,0)$, $\gamma = 2\,\alpha(1,0) - \alpha(1,1)$. Then in the same way we find for all $x \in \mathbb{R}_+^N$ with $x_2 \leq x_1$ the expression

$$\alpha(x) = x_2 \cdot (\alpha(1,1) - \alpha(1,0)) + x_1 \cdot \alpha(1,0) = \beta \cdot (x_1 + x_2) + \gamma \max\{x_1, x_2\}$$

So for $|N| = 2$ and the above β and γ we conclude that $\alpha = \beta\,\alpha_{\text{sum}} + \gamma\,\alpha_{\max}$ on $\mathbb{R}_+^N$. For $N \subset \mathbb{N}$ and $S \subseteq N$ we define $e_S^N \in \mathbb{R}_+^N$ such that $(e_S^N)^k = 1$ if $k \in S$ and $(e_S^N)^k = 0$ for $k \notin S$. Take $n \in \mathbb{N}$. Now, suppose that for all $N \subset \mathbb{N}$ with $|N| \leq n - 1$ it holds that $\alpha = \beta\,\alpha_{\text{sum}} + \gamma\,\alpha_{\max}$ on $\mathbb{R}_+^N$. Without loss of generality, let $N := \{1, 2, \ldots, n\}$ and $\sigma \in \Pi(N)$. Take $x \in C_N^\sigma$. We will now show that, given the regularity of α, $\alpha(x) = \beta\,\alpha_{\text{sum}}(x) + \gamma\,\alpha_{\max}(x)$. We may assume that $x_{\sigma(1)} > 0$, otherwise we can remove agent $\sigma(1)$ from the set N and use the induction hypothesis. But then agent $\sigma(1)$ is one of the smallest non-zero demanders, hence it follows from WMP that

$$\alpha(x) = \alpha(x_{\sigma(1)} e_N^N) - \alpha(x_{\sigma(1)} e_{N\setminus\{\sigma(1)\}}^N) + \alpha((x_{-\sigma(1),0}))$$

Since α is homogeneous, this can be rewritten as

$$\alpha(x) = x_{\sigma(1)}(\alpha(e_N^N) - \alpha(e_{N\setminus\{\sigma(1)\}}^N) + \alpha((x_{-\sigma(1),0}))$$

By WMP it follows that

$$\alpha(x) = x_{\sigma(1)} \left\{ \alpha(e_{N\setminus\{\sigma(1)\}}^N) - \alpha(e_{N\setminus\{\sigma(1),\sigma(2)\}}^N) \right\} + \alpha((x_{-\sigma(1),0}))$$

By the regularity of α we may remove the zero demander $\sigma(1)$ from the profiles

$e^N_{N\setminus\{\sigma(1)\}}, e^N_{N\setminus\{\sigma(1),\sigma(2)\}}$ and $(x_{-\sigma(1)}, 0)$, resulting in

$$\alpha(x) = x_{\sigma(1)} \left\{\alpha(e^{N\setminus\{\sigma(1)\}}_{N\setminus\{\sigma(1)\}}) - \alpha(e^{N\setminus\{\sigma(1)\}}_{N\setminus\{\sigma(1),\sigma(2)\}})\right\} + \alpha(\textstyle\sum_{i\in N\setminus\{\sigma(1)\}} x_{\sigma(i)} e^{N\setminus\{\sigma(1)\}}_{\sigma(i)})$$

Now by our induction hypothesis we have

$$\begin{aligned}
\alpha(x) &= x_{\sigma(1)}\,\beta + \beta\,\textstyle\sum_{i\in N\setminus\{\sigma(1)\}} x_{\sigma(i)} + \gamma\,\max_{i\in N\setminus\{\sigma(1)\}} x_{\sigma(i)} \\
&= \beta\,\textstyle\sum_{i\in N} x_i + \gamma\,\max_{i\in N} x_i \\
&= \beta\,\alpha_{\text{sum}}(x) + \gamma\,\alpha_{\max}(x)
\end{aligned}$$

Then $\alpha(x) = \beta\,\alpha_{\text{sum}}(x) + \gamma\,\alpha_{\max}(x)$ for all $x \in C^\sigma_N$. But note that the definitions of β and γ are independent from the permutation σ. Therefore $\alpha(x) = \beta\,\alpha_{\text{sum}}(x) + \gamma\,\alpha_{\max}(x)$ on $\mathbb{R}^N_+$, since $\bigcup_\sigma C^\sigma_N = \mathbb{R}^N_+$.

$\square$

3 Produce and charge: a dynamic view on cost sharing

Many cost sharing mechanisms known from the literature seem to be static in the sense that they use only the information about costs associated to a finite set of production levels, related to the demand profile of the agents, for a full description of the final cost shares. But a priori there is no reason to omit contingent information about costs for infinite or a continuum of production levels. In this section we will put the cost sharing problem in a more dynamical context and we adopt the approach of Aumann-Shapley pricing. The Aumann-Shapley pricing mechanism for the multi-commodity cost sharing problem considers the marginal costs for a continuum of production levels along a specific production curve towards the aggregate demand. At each level that is attained by the curve, it describes to what extent the different goods can be held responsible for the incurred marginal cost. For one output production technologies the image of the production curve equals the real axis. We take a similar but slightly different dynamical approach by introducing a device that explicitly states to what extent each of the agents participates in the levels of production that are attained towards producing the common minimal satisfactory level of output. Then it makes sense to charge an agent to the same extent for the marginal costs incurred from the different production levels. Such a device specifies at each moment during production of the aggregate demand at which intensity an agent is served. For a cost sharing problem $G = \langle c, \alpha, q^N \rangle \in \mathcal{G}$, we define $N(G) := N$, $\alpha(G) := [0, \alpha(q^N)]$ and $c_G := c$. Denote the unit simplex of $\mathbb{R}_+^N$ by $\Delta(N)$, i.e. the set $\{ x \in \mathbb{R}_+^N \mid \sum_{i \in N} x_i = 1\}$. Then a device that prescribes the intensity of service for the different agents, can be seen as a function on the Cartesian product $\mathcal{G} \times \mathbb{R}_+$, such that

$$\begin{aligned} &\pi(G,t) \in \Delta(N(G)) && \text{if } t \in \alpha(G) \\ &\pi(G,t) = 0 && \text{if } t \in \mathbb{R}_+ \backslash \alpha(G) \end{aligned}$$

We will call such a π a *participation function.*
Let $\mathcal{C}^1 := \{c \in \mathcal{C} \mid c \text{ is continuous and piecewise continuously differentiable}\}$. We restrict our attention to the class of cost sharing problems $\mathcal{G}^1$, consisting of all cost sharing problems G in $\mathcal{G}$ with $c_G \in \mathcal{C}^1$.

We want to express costs associated to any level of output in terms of the marginal costs for production. Since c_G will generally not be differentiable on the whole domain, firstly we extend the derivative c'_G to a Lebesgue-measurable function $Dc_G : \mathbb{R}_+ \to \mathbb{R}_+$

$$Dc_G(t) := \begin{cases} c'_G(t) & \text{if } c \text{ is differentiable at } t \\ 0 & \text{elsewhere} \end{cases}$$

Then as a consequence, the total costs for production of q units of output can be expressed in terms of the *marginal cost* function Dc_G,

$$c_G(q) = \int_0^q Dc_G(t)dt \quad \text{for all } q \in \mathbb{R}_+$$

We make the assumption that participation functions are also well-behaved in the sense that for a fixed cost sharing problem G and $i \in N(G)$ the coordinate mapping $\pi_i(G,t) := \pi(G,t)_i$ is measurable w.r.t. the Borel-Lebesgue measure λ.
It makes sense to charge an agent for the product of the intensity of individual service and the marginal costs of production, i.e. for agent $i \in N(G)$ his cost share is determined by

$$x_i^\pi(G) = \int_{\alpha(G)} \pi_i(G,t) Dc_G(t)dt$$

Note that by our regularity assumptions on c_G we get an efficient allocation[1]. By prescribing the production participating function π we can recover some well-known solution concepts in the cost sharing literature.

For $A \subset \mathbb{R}$ we define the indicator function $I_A : \mathbb{R} \to \{0,1\}$ by

$$I_A(t) := \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \in \mathbb{R} \backslash A \end{cases}$$

In the following examples we will assume that $N = \{1, 2, \ldots, n\}$.

Example 3.1 For α_{sum} and $\pi_i(G,t) := q^i / \sum_{i\in N} q^i$ the outcome $x^\pi(G)$ equals the vector of proportional or average cost shares.

Example 3.2 Assume that the demands are ordered as $q^1 \leq q^2 \leq \ldots \leq q^n$. Define $\pi_i(G,t) := \frac{1}{n} I_{[0,\alpha(q^{1,n})]}(t) + \sum_{k=2}^{i} \frac{1}{n-k+1} I_{(\alpha(q^{k-1,N}),\alpha(q^{k,N})]}(t)$. Then $x^\pi(G)$ is just the serial outcome for G. The formulas for the different cost shares can be understood by the same intuition that is provided by the 'push the button' interpretation of the serial rule (Moulin and Shenker [1992]). Only the way of aggregation may differ.

Example 3.3 Consider the situation where the agents are served in a specific order as given by the permutation $\sigma : N \to N$. Then there are $y_0, y_1, \ldots, y_n \in [0, \alpha(q^N)]$ with $0 = y_0 \leq y_1 \leq \ldots \leq y_n = \alpha(q^N)$ such that $\pi_{\sigma^{-1}(1)}(G,t) = I_{[0,y_1]}(t)$ and $\pi_{\sigma^{-1}(i)}(G,t) = I_{(y_{i-1},y_i]}(t)$ for $i \neq 1$.

Example 3.4 For the cost sharing problem $\langle c, \alpha, q^N \rangle$ with $q^1 \leq q^2 \ldots \leq q^n$ define the intermediate production stages as for the reverse serial rule, $(\bar{x}_k^\alpha)_{k\in N\cup\{0\}}$. In this fashion the reverse serial rule corresponds to dynamic cost sharing with the

[1] We like to indicate that for our results the regularity assumptions on c_G can be weakened. The Lebesgue theorem [1904] about differentiability of monotone real valued functions implies that c_G is differentiable almost everywhere. Now define Dc_G as before. If only it is a Lebesgue-measurable function such that $c_G(q) = \int_0^q Dc_G(t)dt$ for all $q \in \mathbb{R}_+$, then we get efficiency. It is easily seen that at least we have to assume the continuity of c_G for our purposes. However, since it is beyond the scope of this paper, we omit a further detailed discussion.

participation function $\pi_i(G,t) = \frac{1}{n} I_{[0,\bar{x}_1^\alpha]}(t) + \sum_{k=2}^{i} \frac{1}{n+1-k} I_{(\bar{x}_{k-1}^\alpha,\bar{x}_k^\alpha]}(t)$.

4 Consistency in cost sharing problems

In this section we will study the cost sharing problem in the dynamical context as in the former section when the size of the population of agents varies. For the static cost sharing problem as in Section 2 consistency is defined as the desirable stability property for cost sharing mechanisms in case of varying population size. Consistency is defined in terms of the *reduced cost sharing problem.* We indicated in Section 2 that we run into severe difficulties with adapting the cost function for all kinds of aggregation rules. However, those problems can be overcome by considering cost sharing problems in the dynamical context.

Recall the definitions of a production participation function π and corresponding cost sharing mechanism x^{π} (see Section 3). Now we fix a cost sharing problem G in the class $\mathcal{G}^1$, defined in Section 2. Then, what happens to the cost shares of the group of remaining agents $S \subseteq N(G)$ in the situation that agents of $N(G)\backslash S$ leave with prepaying the total of individual cost shares $\sum_{i \in N(G)\backslash S} x_i^{\pi}(G)$? At moment t in the production process, the agents in $N(G)\backslash S$ payed for the fraction $\sum_{i \in N(G)\backslash S} \pi_i(G,t)$ of the marginal costs $Dc_G(t)$. So $\sum_{i \in S} \pi_i(G,t) Dc_G(t)$ is still left over for the remaining agents. In particular, the only marginal costs left to be shared are for those levels of production t where there is a positive participation of the remaining coalition S. The leaving agents pay $\int_0^q \sum_{i \in N(G)\backslash S} \pi_i(G,t) Dc_G(t) dt$ for the first $q \in \alpha(G)$ levels of output. Equivalently, the remaining agents have to pay for the rest,

$$c(q) - \int_0^q \sum_{i \in N(G)\backslash S} \pi_i(G,t) Dc_G(t) dt$$

Further, the agents in coalition S face the reduced cost sharing problem where independently from their own demands, the coalition $N(G)\backslash S$ has to be satisfied. So for the reduced situation the aggregate demand w.r.t. the coalition S is their aggregate demand, given $q^{N(G)\backslash S}$. This means that given the demands of the coalition $N(G)\backslash S$ and the demand profile $q \in \mathbb{R}_+^S$ of the agents in S in the reduced situation, still $\alpha(q, q^{N(G)\backslash S})$ has to be produced. This results in the following natural definition of the reduced cost sharing problem.

Definition 4.1 Let π be a participation function and fix $G \in \mathcal{G}^1$. For $S \subseteq N(G)$ the *reduced cost sharing problem* $G^{\pi,S}$ is the ordered triple $\langle c_G^{\pi,S}, \alpha_G^S, q^S \rangle \in \mathcal{G}^1$ where

$$c_G^{\pi,S}(q) := c_G(q) - \textstyle\int_0^q \sum_{i \in N(G)\backslash S} \pi_i(G,t) Dc_G(t) dt \quad \text{for all } q \in \mathbb{R}_+$$
$$\alpha_G^S(x) = \alpha(x, q^{N(G)\backslash S}) \quad \text{for all } x \in \cup_{N \subseteq \mathbb{N}; |N| < \infty} \mathbb{R}_+^N$$

Now we can define consistency in terms of the reduced cost sharing problem. For a fixed participation function π, the cost sharing mechanism x^{π} is called *consistent* if for all $G \in \mathcal{G}^1$ and all $S \in \mathcal{P}(N(G))$ the cost shares for the agents in S in G equal their individual cost shares determined by x^{π} in the reduced cost sharing problem $G^{\pi,S}$. In general, including also the cost sharing mechanisms for which no description can be used based on a particular participation function, we can use the following definition.

Definition 4.2 Let π be a participation function. The cost sharing mechanism x^π is called *consistent* if for all $G \in \mathcal{G}^1$ and $S \subseteq N(G)$ we have

$$x^\pi(G^{\pi,S}) = x^\pi(G)_{|S}$$

If t belongs to the set of production levels of positive participation of coalition S according to π, then it seems reasonable that an agent $k \in S$ is held responsible for the fraction $\pi_k(G,t)/\sum_{i\in S}\pi_i(G,t)$ of the remaining marginal costs at t. Participating functions π that reckon with this idea give rise to consistent x^π as we will show now.

Theorem 4.3 *Let π be a participation function. Then x^π is consistent if for all $G \in \mathcal{G}^1$ and $S \in \mathcal{P}(N(G))$*

$$\pi_i(G^{\pi,S},t) = \frac{\pi_i(G,t)}{\sum_{k\in S}\pi_k(G,t)} \quad \text{for all } t \in \mathbb{R}_+ \text{ such that } \sum_{k\in S}\pi_k(G,t) > 0$$

Proof: Fix $G \in \mathcal{G}^1$. Suppose that c_G is differentiable at $q \in \mathbb{R}_+$. Then $c_G^{\pi,S}$ is differentiable at q and

$$\begin{aligned}
(c_G^{\pi,S})'(q) &= c_G'(q) - \textstyle\sum_{k\in N(G)\setminus S}\pi_k(G,q)Dc_G(q) \\
&= (1 - \textstyle\sum_{k\in N(G)\setminus S}\pi_k(G,q))c_G'(q) \\
&= \textstyle\sum_{k\in S}\pi_k(G,q)c_G'(q)
\end{aligned}$$

Let A be the set of production levels for which c_G is differentiable. Then if A' is the set of output levels for which there is a positive participation of coalition S, then $Dc_G^{\pi,S}$ is equal to zero outside $\bar{A} := A \cap A'$. Suppose that

$$\pi_i(G^{\pi,S},t) = \frac{\pi_i(G,t)}{\sum_{k\in S}\pi_k(G,t)} \quad \text{for all } t \in \mathbb{R}_+ \text{ with } \sum_{k\in S}\pi_k(G,t) > 0$$

Then this holds especially true on $\bar{A}$, so

$$\begin{aligned}
x_i^\pi(G^{\pi,S}) &= \textstyle\int_{\alpha(G)}\pi_i(G^{\pi,S},t)Dc_G^{\pi,S}(t)dt \\
&= \textstyle\int_{\bar{A}\cap\alpha(G)}\pi_i(G^{\pi,S},t)(c_G^{\pi,S})'(t)dt \\
&= \textstyle\int_{\bar{A}\cap\alpha(G)}\pi_i(G^{\pi,S},t)\sum_{k\in S}\pi_k(G,t)c_G'(t)dt \\
&= \textstyle\int_{\bar{A}\cap\alpha(G)}\pi_i(G,t)c_G'(t)dt \\
&= \textstyle\int_{\alpha(G)}\pi_i(G,t)Dc_G(t)dt \\
&= x_i^\pi(G)
\end{aligned}$$

□

For example, in the dynamic context the (reverse) aggregated serial rule and the generalized proportional rule are consistent. It can be shown that the converse of Theorem 4.3 is not true in general.

5 Cost sharing and cooperative game theory

In this section we will study cost sharing mechanisms as solution concepts for coalitional form games that emerge from cost sharing problems in a natural way. We will focus on the *core* as a solution concept for these games. A topic of future research could be to investigate the significance of other solution concepts for the cooperative games defined in this section.

Recall the definitions of the class of cost sharing problems $\mathcal{G}^1$ and the class of participation functions (Section 3). For a fixed cost sharing problem $G = \langle c, \alpha, q^N \rangle \in \mathcal{G}^1$, the part of the produced amount $\alpha(q^N)$ that agent $i \in N$ participated in according to a fixed participation function π is given by

$$f_i(\pi, G) := \int_0^{\alpha(q^N)} \pi_i(G, t)dt$$

We will refer to $f_i(\pi, G)$ as the *total production participation* of agent i in G according to π. We define, for $x \in [0, \alpha(q^N)]$,

$$\mathrm{W}_G(x) := \sup\{ \int_T Dc_G(t)dt \,|\, T \in \mathcal{B}(\alpha(G)), \lambda(T) = x\}$$

Here $\mathcal{B}([0, \alpha(q^N)])$ denotes the Borel-σ-algebra of the interval $[0, \alpha(q^N)]$. The function W_G reflects the highest or worst cost level that can be achieved for a prescribed amount of output. In this respect, the worst thing that can happen to an agent $i \in N(G)$ is that he is the only one participating in the production process at the most expensive levels, which amounts to the cost share $\mathrm{W}_G(f_i(\pi, G))$. In the same way we can determine the worst possible outcome for any coalition $S \subset N(G)$ participating in total $\sum_{i \in S} f_i(\pi, G)$ units of output, $\mathrm{W}_G(\sum_{i \in S} f_i(\pi, G))$. So for each combination of a cost sharing problem G and a production participation function π, a *pessimistic cost game* arises in a natural way. It consists of an ordered pair $\langle N, k_{\pi,G} \rangle$, where the characteristic function $k_{\pi,G} : \mathcal{P}(N(G)) \to \mathbb{R}_+$ relates each coalition $S \in N(G)$ to its maximal willingness to pay for the satisfactory level of production,

$$k_{\pi,G}(S) = \mathrm{W}_G(\sum_{i \in S} f_i(\pi, G))$$

Example 5.1 Consider a cost sharing problem in $\mathcal{G}^1$ for three agents $N = \{1, 2, 3\}$ where the cost function is defined by $c(q) = (q)^2$, the demand profile is $q^N = (1, 2, 3)$ and the aggregation rule is α_{sum}. Suppose that the participation function π is as in Example 3.1, $\pi_i(G, t) = q^i / \sum_{j \in N} q^j$. Then the total production participation of agent i is given by $\int_0^6 \frac{1}{6} q^i \, dt = q^i$. So the production participation of any coalition S of agents is just $\sum_{i \in S} q^i$. In particular, agent 1 contributes for the production of one unit of the good. The pessimistic value of agent 1 is then the total of marginal costs of processing this unit at the most expensive moments from the first to the last unit, i.e. the incremental cost of processing the last unit $c(q^1 + q^2 + q^3) - c((q^1 + q^2 + q^3) - q^1) = 11$. Similar, for agent 2 the pessimistic value $k_{\pi,G}(2)$ is determined by the cost of processing the last 2 units, $c(6) - c(6 - q^2) = 20$, and for agent 3, $k_{\pi,G}(3) = c(6) - c(6 - q^3) = 27$. For the other coalitions we have $k_{\pi,G}(12) = c(6) - c(6 - (q^1 + q^2)) = 27, k_{\pi,G}(13) = c(6) - c(6 - (q^1 + q^3)) = 32, k_{\pi,G}(23) = c(6) - c(6 - (q^2 + q^3)) = 35$, and $k_{\pi,G}(123) = c(6) - c(6 - 6) = 36$.

The same can be done by defining a function $\mathrm{B}_G : [0, \alpha(q^N)] \to \mathbb{R}_+$, relating a certain output level to the minimal expenses needed for its production. Or more formally, $\mathrm{B}_G(x) = \inf\{ \int_T Dc(t)dt \,|\, T \in \mathcal{B}([0, \alpha(q^N)]), \lambda(T) = x\}$. Then the natural *optimistic cost game* for the combination of the cost sharing problem G and the production participating function π corresponds to the ordered pair $\langle N, k^*_{\pi,G}\rangle$, where the characteristic function $k^*_{\pi,G} : \mathcal{P}(N(G)) \to \mathbb{R}_+$ expresses the minimal obligation of each coalition $S \in N(G)$ to pay for the satisfactory level of production, i.e. the minimal cost associated to the total production participation of coalition S,

$$k^*_{\pi,G}(S) = \mathrm{B}_G(\sum_{i\in S} f_i(\pi, G))$$

We will now give a few classes of cost sharing problems for which there exist simple expressions for the characteristic functions $k_{\pi,G}$ and $k^*_{\pi,G}$, though in general this will not be the case.

Example 5.2 When the cost function $c \in \mathcal{C}^1$ is convex as in Example 5.1, then the marginal costs for production are non-decreasing; the lowest production levels are the cheapest. Then $k_{\pi,G}(S) = c(\alpha(q^N)) - c(\alpha(q^N) - \sum_{i\in S} f_i(\pi, G))$ and $k^*_{\pi,G}(S) = c(\sum_{i\in S} f_i(\pi, G))$.

Example 5.3 For concave $c \in \mathcal{C}^1$ the most expensive spots w.r.t. marginal costs in the production process are right at the start. So in that case $k_{\pi,G}(S) = c(\sum_{i\in S} f_i(\pi, G))$ and $k^*_{\pi,G}(S) = c(\alpha(q^N)) - c(\alpha(q^N) - \sum_{i\in S} f_i(\pi, G))$.

Example 5.4 If α_{sum} is used as aggregation rule then both the optimistic as the pessimistic cost game is independent from the choice of the participation function. More specifically, $k_{\pi,G}(S) = \mathrm{W}_G(\sum_{i\in S} q^i)$ and $k^*_{\pi,G}(S) = \mathrm{B}_G(\sum_{i\in S} q^i)$.

From the properties of the mappings B_G and W_G we learn that the games $k_{\pi,G}$ and $k^*_{\pi,G}$ are monotonic. They are dual in the sense that $k_{\pi,G}(S) = c(\alpha(q^N)) - k^*_{\pi,G}(N\backslash S)$ for all $S \in \mathcal{P}(N)$.

The next lemma proves that for a given desired amount x of output as partial demand of $\alpha(q^N)$, there is a production plan T_x that really generates the highest costs. Especially, this means that the *supremum* in the definition of W_G may be replaced by *maximum*.

Secondly, it is shown that for an additional amount of production beyond the amount x, the most expensive production plan can be found by expansion of the production plan T_x.

Lemma 5.5 *Let $G = \langle c, \alpha, q^N \rangle \in \mathcal{G}^1$. Then for each $x \in [0, \alpha(q^N)]$ there is $T_x \in \mathcal{B}([0, \alpha(q^N)])$ such that $\mathrm{W}_G(x) = \int_{T_x} Dc(t)dt$ and $\lambda(T_x) = x$. The sets can be taken such that $x \leq y \Rightarrow T_x \subseteq T_y$.*

Proof: Take $x \in [0, \alpha(q^N)]$. For $z \in \mathbb{R}_+$ we define $D_z := \{t \in [0, \alpha(q^N)] \mid Dc(t) \geq z\}$. Note that D_z is a measurable set for all z since c is assumed to be continuously differentiable on $[0, \alpha(q^N)]$ except in a finite number of production levels. Let $z(x) := \sup\{z \in \mathbb{R}_+ \mid \lambda(D_z) \geq x\}$. We distinguish between two cases, $\lambda(D_{z(x)}) = x$ and $\lambda(D_{z(x)}) > x$.

First suppose that $\lambda(D_{z(x)}) = x$. We will show that the choice of $T_x := D_{z(x)}$ satisfies. To see this, just take an arbitrary $T \in \mathcal{B}([0, \alpha(q^N)])$ with $\lambda(T) = x$, $T \neq T_x$. Then in particular for $t \in T \backslash T_x$ it holds that $Dc(t) < z(x)$ and therefore

$$\int_{T \backslash T_x} Dc(t)dt \leq z(x) \cdot \lambda(T \backslash T_x) = z(x) \cdot \lambda(T_x \backslash T) \leq \int_{T_x \backslash T} Dc(t)dt$$

As a consequence

$$\begin{aligned} \int_T Dc(t)dt &= \int_{T \cap T_x} Dc(t)dt + \int_{T \backslash T_x} Dc(t)dt \\ &\leq \int_{T \cap T_x} Dc(t)dt + \int_{T_x \backslash T} Dc(t)dt \\ &= \int_{T_x} Dc(t)dt \end{aligned}$$

Therefore $\mathrm{W}_G(x) = \sup\{\int_Y Dc(t)dt \mid \lambda(Y) = x\} = \int_{T_x} Dc(t)dt$.
Now for the second case assume that $\lambda(D_{z(x)}) > x$. This means that

$$\lambda(\{t \in [0, \alpha(q^N)] \mid Dc(t) = z(x)\}) > \lambda(D_{z(x)}) - x$$

Determine $t' \in [0, \alpha(q^N)]$ with $\lambda([0, t'] \cap \{t \in [0, \alpha(q^N)] \mid Dc(t) = z(x)\}) = \lambda(D_{z(x)}) - x$. Now take $T_x := D_{z(x)} \backslash ([0, t'] \cap \{t \in [0, \alpha(q^N) \mid Dc(t) = z(x)\})$. Then $\lambda(T_x) = x$ and the rest is proved analoguously to the first case. Besides, it should be clear from the presented construction that $T_x \subseteq T_y$ whenever $x \leq y$.

$\square$

Lemma 5.6 *Let $G = \langle c, \alpha, q^N \rangle \in \mathcal{G}^1$. Then W_G is concave and B_G is convex.*

Proof: By the equality $\mathrm{W}_G(t) = c(\alpha(q^N)) - \mathrm{B}_G(\alpha(q^N) - t)$ for $t \in [0, \alpha(q^N)]$ it suffices to prove that W_G is concave.
Let $x, y, z \in \mathbb{R}_+$ such that $y \leq z$ and $x + y, x + z \in [0, \alpha(q^N)]$. Take T_x, T_y and T_{x+y} as in the former lemma.

Then by applying the definition of W_G we get

$$\begin{aligned}
\mathrm{W}_G(x+y) - \mathrm{W}_G(y) &= ⨍_{T_{x+y} \backslash T_y} Dc(t)dt \\
&= \sup\{ ⨍_T Dc(t)dt \,|\, T \in \mathcal{B}([0, \alpha(q^N)] \backslash T_y), \lambda(T) = x\} \\
&\geq \sup\{ ⨍_T Dc(t)dt \,|\, T \in \mathcal{B}([0, \alpha(q^N)] \backslash T_z), \lambda(T) = x\} \\
&= ⨍_{T_{x+z} \backslash T_z} Dc(t)dt \\
&= \mathrm{W}_G(x+z) - \mathrm{W}_G(z)
\end{aligned}$$

□

For the further results presented in this section we need some additional terminology. In general a cost game $\langle N, k \rangle$ is said to be *convex* if for all coalitions $S, T \in \mathcal{P}(N)$ it holds that

$$k(S) + k(T) \leq k(T \cup S) + k(S \cap T)$$

Similarly, a cost game $\langle N, k \rangle$ is *concave* if for all coalitions $S, T \in \mathcal{P}(N)$

$$k(S) + k(T) \geq k(S \cup T) + k(S \cap T)$$

Equivalently, $\langle N, k \rangle$ is concave iff for all $i \in N$ and all coalitions $S, T \in \mathcal{P}(N)$ such that $S \subset T \subseteq N \backslash \{i\}$ we have

$$k(S \cup \{i\}) - k(S) \geq k(T \cup \{i\}) - k(T)$$

The *core* of a cost game $\langle N, k \rangle$ is defined as the set of allocations $x \in \mathbb{R}_+^N$ such that for all coalitions $S \in \mathcal{P}(N)$ it holds that $\sum_{i \in S} x_i \leq k(S)$ and $\sum_{i \in N} x_i = k(N)$. We will denote it as $\mathrm{core}(N, k)$. The core elements are all stable in the sense that no coalition can improve by going on its own.

Theorem 5.7 *For all $G \in \mathcal{G}^1$ and all participation functions π, the pessimistic cost game $\langle N, k_{\pi,G} \rangle$ is concave and the optimistic cost game $\langle N, k^*_{\pi,G} \rangle$ is convex.*

Proof: Because of the duality relation between $\langle N, k_{\pi,G} \rangle$ and $\langle N, k^*_{\pi,G} \rangle$, it suffices to prove the concavity of $\langle N, k_{\pi,G} \rangle$. Let $S \subset T \subseteq N \backslash \{j\}$ for some $j \in N$, then $\sum_{i \in S} f_i(\pi, G) \leq \sum_{i \in T} f_i(\pi, G)$. So by the definition of $k_{\pi,G}$ and the concavity of W_G derived in Lemma 5.6 we have

$$\begin{aligned}
k_{\pi,G}(S \cup \{j\}) - k_{\pi,G}(S) &= \mathrm{W}_G(\sum_{i \in S} f_i(\pi, G) + f_j(\pi, G)) - \mathrm{W}_G(\sum_{i \in S} f_i(\pi, G)) \\
&\leq \mathrm{W}_G(\sum_{i \in T} f_i(\pi, G) + f_j(\pi, G)) - \mathrm{W}_G(\sum_{i \in T} f_i(\pi, G)) \\
&= k_{\pi,G}(T \cup \{j\}) - k_{\pi,G}(T)
\end{aligned}$$

□

Recall that for a given participation function π, in Section 3 we defined x^{π} as the cost sharing mechanism that allocates the marginal cost proportional to the individual participations. The next theorem shows that for any participation function π, the mechanism x^{π} generates stable allocations, in terms of the core.

Theorem 5.8 *Let $G \in \mathcal{G}^1$ and π a participation function. Then $x^{\pi}(G)$ is in the core of the game $\langle N, k_{\pi,G} \rangle$.*

Proof: Let $G = \langle c, \alpha, q^N \rangle$. Take $S \in \mathcal{P}(N)$ and let $\beta = \sum_{i \in S} f_i(\pi, G)$. Determine $T_\beta \in \mathcal{B}([0, \alpha(q^N)])$ as in Lemma 5.5 such that $\mathrm{W}_G(\beta) = \int_{T_\beta} Dc(t)dt$. For convenience set $\pi_S(G,t) := \sum_{i \in S} \pi_i(G,t)$ for all t and $T_\beta^c := [0, \alpha(q^N)] \backslash T_\beta$. Then we have for $\delta = \sup\{Dc(t) \mid t \in T_\beta^c\}$,

$$\int_{T_\beta^c} \pi_S(G,t)Dc(t)dt \leq \int_{T_\beta^c} \pi_S(G,t) \cdot \delta dt = \int_{T_\beta} (1-\pi_S(G,t)) \cdot \delta dt \leq \int_{T_\beta} (1-\pi_S(G,t))Dc(t)dt$$

This inequality makes the final part easy,

$$\begin{aligned}
\sum_{i \in S} x_i^{\pi}(G) &= \int_0^{\alpha(q^N)} \pi_S(G,t)Dc(t)dt \\
&= \int_{T_\beta} \pi_S(G,t)Dc(t)dt + \int_{T_\beta^c} \pi_S(G,t)Dc(t)dt \\
&\leq \int_{T_\beta} \pi_S(G,t)Dc(t)dt + \int_{T_\beta} (1 - \pi_S(G,t))Dc(t)dt \\
&= \int_{T_\beta} Dc(t)dt = \mathrm{W}_G(\beta) = k_{\pi,G}(S)
\end{aligned}$$

For the grand coalition the mechanism x^{π} gives an efficient allocation, $\sum_{i \in N} x_i^{\pi} = c(\alpha(q^N)) = k_{\pi,G}(N)$.

□

6 Bankruptcy problems and serial cost sharing

How should a heritage be divided when the heirs' claims sum up to more than there is available? This so called *bankruptcy problem* is the subject of some early Jewish writings contained in the Babylonian Talmud, the extensive collection of Jewish writings on religious and legal decision making in the first five centuries B.C.

Formally, a bankruptcy problem consists of a pair $(E, (d_j)_{j\in N})$, where $E \in (0, \infty)$ is the estate which has to be divided, $N \subset \mathbb{N}$ is a finite set of claimants and $d_i \in [0, \infty)$ is the claim of claimant $i \in N$ such that $\sum_{i\in N} d_i > E$. A *division mechanism* f assigns to each bankruptcy problem $B = (E, (d_j)_{j\in N})$ a division scheme $f(B) \in \mathbb{R}^N_+$, where claimant i gets $f_i(B)$ such that

$$
\begin{aligned}
&1) \quad 0 \leq f_i(B) \leq d_i \text{ for all } i \in N \\
&2) \quad \textstyle\sum_{i\in N} f_i(B) = E
\end{aligned}
$$

The condition $0 \leq f_i(B)$ states that since there is left a positive estate, each claimant should (weakly) profit by it. Furthermore, the inequality $f_i(B) \leq d_i$ $(i \in N)$ assures that no claimant receives more than his claim. The second condition makes sure that the estate is divided efficiently. If there is no estate at all, or $E = 0$, then a division rule f satisfying the above conditions, allocates 0 to each of the claimants. In the following we will be concerned only with non-trivial cases where $E > 0$. In the literature several division mechanisms have been proposed, such as there are the division mechanisms in relation with the early Jewish authors as in O'Neill [1982], the Shapley value and the nucleolus for the associated bankruptcy game (Aumann and Maschler [1985]), and the AP rule (Curiel et al. [1987]). Dagan and Volij [1993] consider each bankruptcy problem as a bargaining problem and propose accessory solution concepts like the RKS solution. We like to mention the following division mechanisms.

Example 6.1 The *Proportional* rule is defined by

$$\mathrm{PROP}_i(E, (d_j)_{j\in N}) := \tfrac{d_i}{\sum_{k\in N} d_k} \cdot E$$

Example 6.2 Determine β as the unique real number such that $\sum_{i\in N} \min\{d_i, \beta\} = E$ Then the *Constraint Equal Award* rule is defined by $\mathrm{CEA}_i(E, (d_j)_{j\in N}) := \min\{d_i, \beta\}$.

Example 6.3 Determine λ as the unique number with $\sum_{i\in N} \max\{d_i - \lambda, 0\} = E$. Then the Constraint Equal Loss rule is given by $\mathrm{CEL}_i(E, (d_i)_{i\in N}) := \max\{d_i - \lambda, 0\}$.

The class of bankruptcy problems can be embedded into the class of cost sharing problems by considering $(d_j)_{j\in N}$ as the demand profile of the agents and $\sum_{i\in N} d_i - E$ as the deficit or cost to be shared. So we see the bankruptcy problem $(E, (d_j)_{j\in N})$ as the cost sharing problem $\langle c_E, \alpha_{\text{sum}}, (d_j)_{j\in N} \rangle$ where $c_E : [0, \infty) \to [0, \infty)$ is the cost function defined by $c_E(x) := \max\{x - E, 0\}$ for all $x \in [0, \infty)$ and $(d_j)_{j\in N}$ is the profile of demands. Especially, when $x \leq E$ then $c(x) = 0$ as there is no deficit. As a result each cost sharing method can be used as a solution concept for bankruptcy problems. Some of the well-known cost sharing mechanisms are even closely connected to solution concepts for bankruptcy problems that have been studied in the literature. For example, it is easily checked that $\mathrm{PROP}_i(E, (d_j)_{j\in N}) = d_i - \xi_i^p(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N})$ for all $i \in N$. In the same way the CEA rule (CEL rule) corresponds to the serial cost sharing mechanism (reverse serial mechanism).

Theorem 6.4 *Let $(E, (d_j)_{j\in N})$ be a bankruptcy problem. Then for all $i \in N$*

$$\mathrm{CEA}_i(E, (d_j)_{j\in N}) = d_i - \xi_i^{\mathrm{s}}(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N})$$

Proof: Let β be such that $\sum_{i\in N} \min\{d_i, \beta\} = E$. Without loss of generality we assume that $E > 0$, $N = \{1, 2, \ldots, n\}$ and that the claims can be ordered such that $d_1 \leq d_2 \leq \ldots \leq d_n$. Then the serial cost share for agent $i \in N$ in the associated cost sharing problem is determined by

$$\xi_i^{\mathrm{s}}(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N}) = \sum_{j=1}^{i} \frac{c_E(x_j) - c_E(x_{j-1})}{n - j + 1}$$

Here for $j \in \{1, 2, \ldots, n\}$, x_j denotes the intermediate production level defined by

$$\begin{aligned} x_0 &:= 0 \\ x_j &:= \textstyle\sum_{r=1}^{j-1} d_r + (n + 1 - j) \cdot d_j \end{aligned}$$

For convenience set $d_0 := 0$. Let $k \in \{1, 2, \ldots, n\}$ be the smallest number such that $\beta \in [d_{k-1}, d_k]$. Then $E = \sum_{j=1}^{k-1} d_j + (n + 1 - j) \cdot \beta$ and

$$c_E(x_j) = \begin{cases} 0 \quad \text{for } j < k \\ \sum_{i=k}^{j-1}(d_i - \beta) + (n + 1 - j) \cdot (d_j - \beta) \quad \text{for } j \geq k \end{cases}$$

Thus for $i < k$ we have $\xi_i^{\mathrm{s}}(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N}) = 0$, while $\mathrm{CEA}_i(E, (d_j)_{j\in N}) = d_i$. For $i \geq k$ we have $\mathrm{CEA}_i(E, (d_j)_{j\in N}) = \beta$ and

$$\begin{aligned} \xi_i^{\mathrm{s}}(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N}) &= \textstyle\sum_{j=k}^{i} \frac{c_E(x_j) - c_E(x_{j-1})}{n+1-j} \\ &= (d_k - \beta) + (d_{k+1} - d_k) + \ldots + (d_i - d_{i-1}) \\ &= d_i - \beta \end{aligned}$$

So for all $i \in N$ we have $\xi_i^{\mathrm{s}}(c_E, \alpha_{\mathrm{sum}}, (d_j)_{j\in N}) = d_i - \mathrm{CEA}_i(E, (d_j)_{j\in N})$.

$\square$

The CEA division mechanism can be characterized in a similar way as the serial cost sharing mechanism (Theorem 2.9) using similar characterizing properties.

Definition 6.5 A division mechanism f satisfies *Independence of Size of Larger Claims* (ISLC) if for all bankruptcy problems (E, d^N) it holds that for all $i, j \in N$,

$$\{d_j \geq d_i \text{ and } \tilde{d}_j \geq d_i\} \Rightarrow f_i(E, d^N) = f_i(E, (d^N_{-j}, \tilde{d}_j))$$

Definition 6.6 A division mechanism f satisfies *Equal Treatment of Equals* (ETE) if for all bankruptcy problems (E, d^N) it holds that $f_i(E, d^N) = f_j(E, d^N)$ whenever $d_i = d_j$ for $i, j \in N$.

Theorem 6.7 *The* CEA *rule is the unique division mechanism satisfying* ETE *and* ISLC.

Proof: Suppose f is a division mechanism with the properties ETE and ISLC. Then $\tilde{f}(c_E, \alpha_{\text{sum}}, d^N) := d^N - f(E, d^N)$ defines a cost sharing mechanism on the subclass of those cost sharing problems related to bankruptcy problems It satisfies the properties EFF, ETE and ISLD. Therefore, it is the restriction of the serial mechanism (see Remark 2.10). So $\xi^{\text{s}}(c_E, \alpha_{\text{sum}}, d^N) = d^N - f(E, d^N)$ for all bankruptcy problems (E, d^N). Then by Theorem 6.4, $f(E, d^N) = \text{CEA}(E, d^N)$.

□

In the same way as in Theorem 6.4 we can relate the CEL rule to the reverse serial rule.

Theorem 6.8 *Let $\langle E, (d_j)_{j\in N}\rangle$ be a bankruptcy problem. Then for all $i \in N$*

$$\text{CEL}_i(E, (d_j)_{j\in N}) = d_i - \xi^{\text{rs}}(c_E, \alpha_{\text{sum}}, (d_j)_{j\in N})$$

Proof: Analogous to that of Theorem 6.4.

□

Let $B = (E, (d_i)_{i\in N})$ be a bankruptcy problem and G_B the related cost sharing problem. The worst outcome for a coalition $S \in \mathcal{P}(N)$ is the one where the members of $N\backslash S$ have 'first choice', i.e. the coalition $N\backslash S$ just takes the total amount of their claims up to the maximum E and leaves. Then S gets the remainder. In this view we define the *pessimistic bankruptcy game* $\langle N, v_p^B\rangle$, where the characteristic function is defined by $v_p^B(S) = \max\{0, E - \sum_{i\in N\backslash S} d_i\}$ for all $S \in \mathcal{P}(N)$. It is easily seen that $\langle N, v_p^B\rangle$ is related to $\langle N, k_{G_B}\rangle$ via $v_p^B(S) = \sum_{i\in S} d_i - k_{G_B}(S)$ for all $S \in \mathcal{P}(N)$.

For the cost sharing problems arising from bankruptcy problems we can characterize the core of the pessimistic cost game by characterizing the core of the pessimistic bankruptcy game. The core of the pessimistic bankruptcy game is generated by the different division mechanisms (see Curiel et al. [1987]).

7 Concluding remarks

Moulin and Shenker [1992] and Moulin [1993] discuss a natural way to relate the combination of a cost sharing problem and a cost sharing mechanism to a normal form game in case the agents are endowed with preferences over the consumption vectors. In situations where there are decreasing returns to scale the strategic features of this normal form game make the use of the serial mechanism compelling. It could be a topic for future research to investigate for what kind of aggregation rules similar results hold.

Nowadays there is a growing interest in the problem of extending the serial cost sharing mechanism to multi-commodity situations. In this respect we can mention Kolpin [1994], Friedman and Moulin [1995] and Koster et al. [1996]. The proposals all take the sum as the aggregation rule for the different outputs of the production facility. Maybe the dynamic approach to cost sharing models for general demand aggregation could play an interesting role in catching the more realistic situations, where the demand for the different outputs is aggregated differently. For instance, like in Example 2.3 imagine that each output commodity has its own characteristic loss function. Then though the basic aggregation might be α_{sum}, the losses are compensated in a different way.

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